Supplementary material to
Human motion estimation on Lie groups using IMU measurements

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Derivation of $\mathcal{H}_{k+1}$ given accelerometer measurements

The full state of the system X_k is of the form

$$\begin{aligned} X_k &= \text{blkdiag}\{\theta_k, \omega_k, \alpha_k\}, . \\ \theta_k &= \text{blkdiag}\{\theta_k^1, ..., \theta_k^m\}, \\ \omega_k &= \text{blkdiag}\{\omega_k^1, ..., \omega_k^m\}, \\ \alpha_k &= \text{blkdiag}\{\alpha_k^1, ..., \alpha_k^m\}, \end{aligned}$$

where subscript k denotes time instant, θ_k^i is position of the i-th joint, ω_k^i is velocity of the i-th joint, α_k^i is acceleration of the i-th joint, and m is the number of joints of a body. Measurement Jacobian $\mathcal{H}_{k+1}$ relating the accelerometer measurement and joint l is given as

$$\begin{bmatrix} \mathcal{H}_{k+1}^l \\ 1 \end{bmatrix} = \frac{\partial \mathcal{K}_0^{s,R}}{\partial X_{k+1|k}^l} \left(\ddot{p}_{k+1|k} + g \right) + \mathcal{K}_0^{s,R} \frac{\partial \ddot{p}_{k+1|k}}{\partial X_{k+1|k}^l}, \tag{1}$$

where $\mathcal{K}_j^{i,R}$ stands for the rotational component of the forward kinematics between the i-th and j-th joints (alternatively 0 represents origin, and s denotes sensor), $\ddot{p}_{k+1|k}$ represents an acceleration of the sensor s represented in the base frame and given in homogeneous coordinates, while g is the gravity vector in homogeneous coordinates. The subscript $k+1|k$ denotes prediction at time instant $k+1$ given the measurement up to and including time instant k. In order to evaluate (1) we need to compute partial derivatives of $\mathcal{K}_0^{s,R}$ and $\ddot{p}_{k+1|k}$ with respect to position, velocity, and acceleration of the full system state for joint l, i.e., $X_{k+1|k}^l$.

A. *Positional part*

Here we consider the evaluation of $\mathcal{H}_{k+1}^l$ with respect to position $\theta_{k+1|k}^l$. We start by evaluating the partial derivative of forward kinematics $\mathcal{K}_0^{s,R}$ with respect to the positional variable $\theta_{k+1|k}^{l,r}$, where r relates to the r-th generator, $r = 1, .., d_l$, with d_l being the number of degrees of freedom of joint l. This evaluates to

$$\frac{\partial \mathcal{K}_0^{s,R}}{\partial \theta_{k+1|k}^{l,r}} = \mathcal{K}_0^{l,R} \theta_{k+1|k}^{l,r} E^r \mathcal{K}_l^{s,R}, \tag{2}$$

where $E^{l,r}$ represents the r-th generator of a Lie group representing the l-th joint. The evaluation of the partial derivative of acceleration $\ddot{p}_{k+1|k}$ with respect to the positional variable $\theta_{k+1|k}^{l,r}$ evaluates to

$$\begin{aligned} \frac{\partial \ddot{p}_{k+1|k}}{\partial \theta_{k+1|k}^{l,r}} = & \sum_{i=1}^{n} \left(\sum_{j=1}^{i} \begin{Bmatrix} \mathcal{K}_l^0 \, E^{l,r} \, \mathcal{K}_j^l \, S_{k+1|k}^{j,\omega} \, \mathcal{K}_i^j \, S_{k+1|k}^{i,\omega} \, \mathcal{K}_s^i \, , & l \le j \\ \mathcal{K}_j^0 \, S_{k+1|k}^{j,\omega} \, \mathcal{K}_l^j \, E^{l,r} \, \mathcal{K}_i^l \, S_{k+1|k}^{i,\omega} \, \mathcal{K}_s^i \, , & j < l \le i \\ \mathcal{K}_j^0 \, S_{k+1|k}^{j,\omega} \, \mathcal{K}_i^j \, S_{k+1|k}^{i,\omega} \, \mathcal{K}_l^i \, E^{l,r} \, \mathcal{K}_s^l \, , & i < l \end{Bmatrix} \right) \mathcal{O} + \\ & \sum_{i=1}^{n} \left(\sum_{j=i+1}^{n} \begin{Bmatrix} \mathcal{K}_l^0 \, E^{l,r} \, \mathcal{K}_i^l \, S_{k+1|k}^{i,\omega} \, \mathcal{K}_j^i \, S_{k+1|k}^{j,\omega} \, \mathcal{K}_s^j \, , & l \le i \\ \mathcal{K}_i^0 \, S_{k+1|k}^{i,\omega} \, \mathcal{K}_l^i \, E^{l,r} \, \mathcal{K}_j^l \, S_{k+1|k}^{j,\omega} \, \mathcal{K}_s^j \, , & i < l \le j \\ \mathcal{K}_i^0 \, S_{k+1|k}^{i,\omega} \, \mathcal{K}_j^i \, S_{k+1|k}^{j,\omega} \, \mathcal{K}_l^j \, E^{l,r} \, \mathcal{K}_s^l \, , & j < l \end{Bmatrix} \right) \mathcal{O} + \\ & \sum_{i=1}^{n} \begin{Bmatrix} \mathcal{K}_l^0 \, E^{l,r} \, \mathcal{K}_i^l \, S_{k+1|k}^{i,\alpha} \, \mathcal{K}_s^i \, , & l \le i \\ \mathcal{K}_i^0 \, S_{k+1|k}^{i,\alpha} \, \mathcal{K}_l^i \, E^{l,r} \, \mathcal{K}_s^l \, , & i < l \end{Bmatrix} \mathcal{O} \, , \end{aligned} \tag{3}$$

This work has been supported from the Unity Through Knowledge Fund under the project *Cooperative Cloud based Simultaneous Localization and Mapping in Dynamic Environments* (cloudSLAM) and by the Ministry of Science, Education and Sports of the Republic of Croatia under the grant *Center of Research Excellence for Data Science and Cooperative Systems* (CoE ACROSS).

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where

$$S^{i,\omega}_{k+1|k} = \sum_{r=1}^{d_i} \left(\omega^{i,r}_{k+1|k} E^{i,r} \right) , \tag{4}$$

which is a function of the number of degrees of freedom d_i of the i-th joint, and the superscript ω denotes that the velocity components are summed up. The three parts in (3) arise from evaluating partial derivatives of the three components existing in equation (27) of the original manuscript, i.e., the two centripetal components and the joint acceleration component. Depending on the location within kinematic chain of the considered joint l, different terms need to be applied. However, this is still a direct result of evaluating partial derivatives of (27) of the original manuscript. The complete positional component can now be calculated as

$$\begin{bmatrix} \mathcal{H}^{\theta,l,r}_{k+1} \\ 1 \end{bmatrix} = \frac{\partial \mathcal{K}^{s,R}_0}{\partial \theta^{l,r}_{k+1|k}} \left(\ddot{p}_{k+1|k} + g \right) + \mathcal{K}^{s,R}_0 \frac{\partial \ddot{p}_{k+1|k}}{\partial \theta^{l,r}_{k+1|k}} . \tag{5}$$

B. Velocity part

Since $\mathcal{K}^{s,R}_0$ is only a function of the joint position $\theta^{l}_{k+1|k}$, the partial derivative of forward kinematics with respect to the velocity component is

$$\frac{\partial \mathcal{K}^{s,R}_0}{\partial \omega^{l,r}_{k+1|k}} = 0 . \tag{6}$$

We now evaluate the partial derivative of acceleration $\ddot{p}_{k+1|k}$, with respect to the velocity variable $\omega^{l,r}_{k+1|k}$, which evaluates to the following expression

$$\begin{aligned} \frac{\partial \ddot{p}_{k+1|k}}{\partial \omega^{l,r}_{k+1|k}} = \mathcal{K}^0_l E^{l,r} \sum_{i=l}^{n} \left(\mathcal{K}^l_i S^{i,\omega}_{k+1|k} \mathcal{K}^i_s \right) \mathcal{O} \; + \; \sum_{j=1}^{l} \left(\mathcal{K}^0_j S^{j,\omega}_{k+1|k} \mathcal{K}^j_l \right) E^{l,r} \mathcal{K}^l_s \mathcal{O} \; + \\ \mathcal{K}^0_l E^{l,r} \sum_{j=l+1}^{n} \left(\mathcal{K}^l_j S^{j,\omega}_{k+1|k} \mathcal{K}^j_s \right) \mathcal{O} \; + \; \sum_{i=1}^{l-1} \left(\mathcal{K}^0_i S^{i,\omega}_{k+1|k} \mathcal{K}^i_l \right) E^{l,r} \mathcal{K}^l_s \mathcal{O} . \end{aligned} \tag{7}$$

The four parts of this derivative arise from the two centripetal force components (two per each) given in equation (27) of the original manuscript. The complete velocity component can now be calculated as

$$\begin{bmatrix} \mathcal{H}^{\omega,l,r}_{k+1} \\ 1 \end{bmatrix} = \frac{\partial \mathcal{K}^{s,R}_0}{\partial \omega^{l,r}_{k+1|k}} \left(\ddot{p}_{k+1|k} + g \right) + \mathcal{K}^{s,R}_0 \frac{\partial \ddot{p}_{k+1|k}}{\partial \omega^{l,r}_{k+1|k}} . \tag{8}$$

C. Acceleration part

Here, we evaluate the acceleration term. Since $\mathcal{K}^{s,R}_0$ is only function of the joint position $\theta^{l}_{k+1|k}$, the partial derivative of forward kinematics with respect to the acceleration component is

$$\frac{\partial \mathcal{K}^{s,R}_0}{\partial \alpha^{l,r}_{k+1|k}} = 0 . \tag{9}$$

The partial derivative of acceleration $\ddot{p}_{k+1|k}$ with respect to the r-th component of the acceleration of the l-th joint, $\alpha^{l,r}_{k+1|k}$, evaluates as

$$\frac{\partial \ddot{p}_{k+1|k}}{\partial \alpha^{l,r}_{k+1|k}} = \mathcal{K}^0_l E^{l,r} \mathcal{K}^l_s \mathcal{O} . \tag{10}$$

This derivative arise from the joint acceleration component given in equation (27) of the original manuscript. The complete acceleration component can now be calculated as

$$\begin{bmatrix} \mathcal{H}^{\alpha,l,r}_{k+1} \\ 1 \end{bmatrix} = \frac{\partial \mathcal{K}^{s,R}_0}{\partial \alpha^{l,r}_{k+1|k}} \left(\ddot{p}_{k+1|k} + g \right) + \mathcal{K}^{s,R}_0 \frac{\partial \ddot{p}_{k+1|k}}{\partial \alpha^{l,r}_{k+1|k}} . \tag{11}$$

Finally, the full $\mathcal{H}_{k+1}$ relating sensor measurement and the system variables associated to m joints is constructed as

$$\mathcal{H}^{l}_{k+1} = \begin{bmatrix} \mathcal{H}^{\theta,l}_{k+1} & \mathcal{H}^{\omega,l}_{k+1} & \mathcal{H}^{\alpha,l}_{k+1} \end{bmatrix} . \tag{12}$$